

## INFINITE PRIMITIVE DIRECTED GRAPHS

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ABSTRACT. A group  $G$  of permutations of a set  $\Omega$  is *primitive* if it acts transitively on  $\Omega$ , and the only  $G$ -invariant equivalence relations on  $\Omega$  are the trivial and universal relations. A graph  $\Gamma$  is *primitive* if its automorphism group acts primitively on its vertex set.

A graph  $\Gamma$  has *connectivity one* if it is connected and there exists a vertex  $\alpha$  of  $\Gamma$ , such that the induced graph  $\Gamma \setminus \{\alpha\}$  is not connected. If  $\Gamma$  has connectivity one, a *block* of  $\Gamma$  is a connected subgraph that is maximal subject to the condition that it does not have connectivity one.

The primitive undirected graphs with connectivity one have been fully classified by Jung and Watkins: the blocks of such graphs are primitive, pairwise-isomorphic and have at least three vertices. When one considers the general case of a directed primitive graph with connectivity one, however, this result no longer holds. In this paper we investigate the structure of these directed graphs, and obtain a complete characterisation.

## 1. PRELIMINARIES

Throughout this note, a *graph* will be a directed graph without multiple edges or loops. A graph  $\Gamma$  will be thought of as a pair  $(V\Gamma, E\Gamma)$ , where  $V\Gamma$  is the set of *vertices* and  $E\Gamma$  the set of *edges* of  $\Gamma$ . The set  $E\Gamma$  consists of ordered pairs of distinct elements of  $V\Gamma$ . Two vertices  $\alpha, \beta \in V\Gamma$  are *adjacent* if either  $(\alpha, \beta)$  or  $(\beta, \alpha)$  lies in  $E\Gamma$ . All paths will be undirected, unless otherwise stated. A graph is *infinite* if its vertex set is infinite.

Two vertices are *connected* if there exists an undirected path in  $\Gamma$  between them, while a graph is *connected* if any two vertices are connected. The *distance* between two connected vertices  $\alpha$  and  $\beta$  in  $\Gamma$

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is the length of the shortest path between them, and is denoted by  $d_\Gamma(\alpha, \beta)$ .

We shall be deducing the properties of directed graphs from characterisations of *undirected graphs*. This, of course, requires us to carefully define the latter in a way that preserves the natural relationship between the two. The term *undirected graph* will henceforth refer to a directed graph with the property that, whenever  $(\alpha, \beta) \in E\Gamma$ , we have  $(\beta, \alpha) \in E\Gamma$ . In this case, it is sometimes convenient to replace each pair of edges  $(\alpha, \beta)$  and  $(\beta, \alpha)$  with the unordered pair  $\{\alpha, \beta\}$ .

Groups, and in particular groups of automorphisms, will play a leading role in many of the arguments presented herein. Throughout this work,  $G$  will be a group of permutations of an infinite set  $\Omega$ , where  $\Omega$  will usually be the vertex set of some infinite graph.

If  $\alpha \in \Omega$  and  $g \in G$ , we denote the image of  $\alpha$  under  $g$  by  $\alpha^g$ . Following this notation, all permutations will act on the right. The set of images of  $\alpha$  under all elements of  $G$  is called an *orbit* of  $G$ , and is denoted by  $\alpha^G$ . There is a natural action of  $G$  on the  $n$ -element subsets and  $n$ -tuples of  $\Omega$  via  $\{\alpha_1, \dots, \alpha_n\}^g := \{\alpha_1^g, \dots, \alpha_n^g\}$  and  $(\alpha_1, \dots, \alpha_n)^g := (\alpha_1^g, \dots, \alpha_n^g)$  respectively.

If  $\alpha \in \Omega$ , we denote the stabiliser of  $\alpha$  in  $G$  by  $G_\alpha$ , and if  $\Sigma \subseteq \Omega$  we denote the setwise and pointwise stabilisers of  $\Sigma$  in  $G$  by  $G_{\{\Sigma\}}$  and  $G_{(\Sigma)}$  respectively.

The group  $G$  is *transitive* on  $\Omega$  if  $G$  has one orbit on  $\Omega$ , namely  $\Omega$  itself. A transitive group  $G$  is said to act *regularly* on  $\Omega$  if  $G_\alpha = 1$  for each  $\alpha \in \Omega$ .

A  $G$ -congruence on  $\Omega$  is an equivalence relation  $\approx$  on  $\Omega$  satisfying

$$\alpha \approx \beta \Leftrightarrow \alpha^g \approx \beta^g,$$

for all  $\alpha, \beta \in \Omega$  and  $g \in G$ . A transitive group  $G$  is *primitive* on  $\Omega$  if the only  $G$ -congruencies admitted by  $\Omega$  are the trivial and universal equivalence relations. The following is well known.

**Theorem 1.1.** ([1, Theorem 4.7]) *If  $G$  is a transitive group of permutations on  $\Omega$ , and  $|\Omega| > 1$ , then  $G$  is primitive on  $\Omega$  if and only if, for every  $\alpha \in \Omega$ , the stabiliser  $G_\alpha$  is a maximal subgroup of  $G$ .  $\square$*

A permutation  $\sigma$  of  $V\Gamma$  is an *automorphism* of  $\Gamma$  if it preserves the edge-structure of  $\Gamma$ ; that is,

$$e \in E\Gamma \Leftrightarrow e^\sigma \in E\Gamma.$$

The set of all automorphisms of the graph  $\Gamma$  form a group called the *automorphism group* of  $\Gamma$ , denoted by  $\text{Aut } \Gamma$ . A graph is *primitive* if  $\text{Aut } \Gamma$  is primitive on the set  $V\Gamma$ , and is *automorphism-regular* if  $\text{Aut } \Gamma$  acts regularly on  $V\Gamma$ .

The following theorem due to D. G. Higman gives a test for primitivity.

**Theorem 1.2.** ([2]) *A transitive group of permutations of a set  $\Omega$  is primitive if and only if every graph with vertex set  $\Omega$  and edge-set  $(\alpha, \beta)^G$  is connected whenever  $\alpha$  and  $\beta$  are distinct elements of  $\Omega$ .  $\square$*

A useful consequence of this result is that every primitive graph must be connected.

The *connectivity* of an infinite connected graph  $\Gamma$  is the smallest possible size of a subset  $W$  of  $V\Gamma$  for which the induced graph  $\Gamma \setminus W$  is disconnected. A *block* of  $\Gamma$  is a connected subgraph that is maximal subject to the condition it has connectivity strictly greater than one. If  $\Gamma$  has connectivity one, then the vertices  $\alpha$  for which  $\Gamma \setminus \{\alpha\}$  is disconnected are called the *cut vertices* of  $\Gamma$ .

A group acting on a graph  $\Gamma$  is said to be *vertex-transitive* or *edge-transitive* if it acts transitively on the set of vertices or edges of  $\Gamma$  respectively. Similarly, a graph admitting such a group will be referred to as being *vertex-* or *edge-transitive*.

## 2. LOCAL STRUCTURE

Consider the following construction. Let  $V_1$  be the set of cut vertices of a connected graph  $\Gamma$ , and let  $V_2$  be a set in bijective correspondence with the set of blocks of  $\Gamma$ . We let  $T$  be a bipartite graph whose parts are  $V_1$  and  $V_2$ . Two vertices  $\alpha \in V_1$  and  $x \in V_2$  are adjacent in  $T$  if and only if  $\alpha$  is contained in the block of  $\Gamma$  corresponding to  $x$ ; in this case, there are two edges in  $T$  between the vertices  $\alpha$  and  $x$ , one going in each direction. Thus,  $T$  can be considered to be an undirected graph. In

fact, this construction yields a tree, which is called the *block-cut-vertex tree* of  $\Gamma$ . Note that if  $\Gamma$  has connectivity one and block-cut-vertex tree  $T$ , then any group  $G$  acting on  $\Gamma$  has a natural action on  $T$ .

Let  $\Gamma$  be a primitive graph with connectivity one whose blocks have at least three vertices, and suppose  $G$  is a vertex- and edge-transitive group of automorphisms of  $\Gamma$ . Since  $\Gamma$  is a vertex-transitive graph with connectivity one, every vertex is a cut vertex. Fix some block  $\Lambda$  of  $\Gamma$ , and let  $H$  be the subgroup of the automorphism group  $\text{Aut } \Lambda$  induced by the setwise stabiliser  $G_{\{\Lambda\}}$  of  $V\Lambda$  in  $G$ . Let  $T$  be the block-cut-vertex tree of  $\Gamma$ , and let  $x$  be the vertex of  $T$  that corresponds to the block  $\Lambda$ . Our aim in this section is to show  $H$  is primitive but not regular.

If  $x_1$  and  $x_2$  are distinct vertices of the tree  $T$ , we use  $C(T \setminus \{x_1\}, x_2)$  to denote the connected component of  $T \setminus \{x_1\}$  that contains the vertex  $x_2$ .

**Lemma 2.1.** *If  $G$  acts primitively on the vertices of  $\Gamma$ , then the group  $H$  acts primitively on the vertices of  $\Lambda$ .*

*Proof.* Fix  $\alpha \in V\Lambda$  and suppose, for a contradiction, the group  $H$  does not act primitively on  $V\Lambda$ . Then there is a vertex  $\gamma \in V\Lambda \setminus \{\alpha\}$  such that the graph  $\Lambda' := (V\Lambda, (\alpha, \gamma)^H)$  is not connected. Let  $\Gamma' := (V\Gamma, (\alpha, \gamma)^G)$ . We will show this graph cannot be connected, and hence  $G$  cannot be primitive.

Let  $\{\Delta_i\}_{i \in I}$  be the set of connected components of  $\Lambda'$  and let

$$\mathcal{C}_i := \bigcup_{\delta \in \Delta_i} C(T \setminus \{x\}, \delta) \cap V\Gamma.$$

Suppose  $\delta_i \in \mathcal{C}_i$  and  $\delta_j \in \mathcal{C}_j$ , with  $i \neq j$ . We claim  $\delta_i$  and  $\delta_j$  are not adjacent in  $\Gamma'$ . Indeed, since the distance  $d_T(\alpha, \gamma)$  between  $\alpha$  and  $\gamma$  in  $T$  is equal to 2, if  $\delta_i$  and  $\delta_j$  are to be adjacent in the edge-transitive graph  $\Gamma'$ , it must be the case that  $d_T(\delta_i, \delta_j) = 2$ . If either  $\delta_i$  or  $\delta_j$  is not adjacent to  $x$  in  $T$  then  $d_T(\delta_i, \delta_j) > 2$ , so they cannot be adjacent in  $\Gamma'$ . On the other hand, if  $\delta_i$  and  $\delta_j$  are adjacent to  $x$  in  $T$ , then they both lie in  $V\Lambda = V\Lambda'$ , and therefore  $\delta_i \in \Delta_i$  and  $\delta_j \in \Delta_j$ . In this case, if they are adjacent in  $\Gamma'$  then there exists  $g \in G$  such that either  $(\delta_i, \delta_j)$  or  $(\delta_j, \delta_i)$  is equal to  $(\alpha, \gamma)^g$ . Such an automorphism must fix

$V\Lambda$  setwise, and therefore lies in  $G_{\{\Lambda\}}$ . Thus, there exists an element  $h \in H$  such that either  $(\delta_i, \delta_j)$  or  $(\delta_j, \delta_i)$  is equal to  $(\alpha, \gamma)^h$ , meaning that  $\delta_i$  and  $\delta_j$  are adjacent in  $\Lambda'$ ; however, this contradicts the fact that  $\delta_i$  and  $\delta_j$  are in distinct components of  $\Lambda'$ . Hence,  $\delta_i$  and  $\delta_j$  are not adjacent in  $\Gamma'$ .

Therefore, there can be no path in  $\Gamma'$  between a vertex in  $\mathcal{C}_i$  and a vertex in  $\mathcal{C}_j$  whenever  $i \neq j$ , and so the graph  $\Gamma'$  is not connected. Whence,  $G$  cannot act primitively on  $V\Gamma$  by Theorem 1.2.  $\square$

Fix distinct vertices  $\alpha, \beta \in V\Gamma$  and recall that  $\alpha$  and  $\beta$  are also vertices of the block-cut-vertex tree  $T$ .

A *geodesic* between two vertices is a shortest path between them. In a tree, there is a unique geodesic between any two vertices. Let  $[\alpha, \beta]_T$  be the  $T$ -geodesic between  $\alpha$  and  $\beta$ , and let  $(\alpha, \beta)_T$  be the  $T$ -geodesic  $[\alpha, \beta]_T$  excluding both  $\alpha$  and  $\beta$ . This notation extends obviously to  $[\alpha, \beta)_T$  and  $(\alpha, \beta]_T$ .

Since  $\alpha$  and  $\beta$  are vertices of both  $\Gamma$  and  $T$ , the distance  $d_T(\alpha, \beta)$  is even, so we may choose a vertex  $y \in (\alpha, \beta)_T$  that is distinct from  $\alpha$  and  $\beta$ .

**Lemma 2.2.** *If  $g \in G_\alpha$  does not fix  $y \in VT$ , and  $\delta \notin C(T \setminus \{y\}, \alpha)$ , then  $\delta^g \notin C(T \setminus \{y\}, \beta)$ . Similarly, if  $g \in G_\beta$  does not fix  $y$  and  $\delta \notin C(T \setminus \{y\}, \beta)$  then  $\delta^g \notin C(T \setminus \{y\}, \alpha)$ .*

*Proof.* If  $\delta \notin C(T \setminus \{y\}, \alpha)$  and  $\delta^g \in C(T \setminus \{y\}, \beta)$  then  $\delta, \delta^g \notin C(T \setminus \{y\}, \alpha)$ , so we must have  $g \in G_{\alpha, y}$ . Similarly, if  $\delta \notin C(T \setminus \{y\}, \beta)$  and  $\delta^g \in C(T \setminus \{y\}, \alpha)$  then  $\delta, \delta^g \notin C(T \setminus \{y\}, \beta)$ , so we must have  $g \in G_{\beta, y}$ .  $\square$

**Lemma 2.3.** *If  $g \in G_\alpha$  does not fix the vertex  $y$  and  $\delta \notin C(T \setminus \{y\}, \alpha)$  then  $d_T(y, \delta^g) > d_T(y, \delta)$ . Similarly, if  $g \in G_\beta$  does not fix  $y$  and  $\delta \notin C(T \setminus \{y\}, \beta)$  then  $d_T(y, \delta^g) > d_T(y, \delta)$ .*

*Proof.* Let  $y'$  be the vertex adjacent to  $y$  in  $[\alpha, y]_T$ . If  $\delta \notin C(T \setminus \{y\}, \alpha)$  then  $y \in [\alpha, \delta]_T$ . Since  $g \in G_\alpha \setminus G_y$ , both  $y$  and  $y'$  lie on the geodesic  $[\delta, \delta^g]_T$ , with  $y' \in [\delta^g, y]_T$ . Thus  $d_T(\delta^g, y) = d_T(\delta^g, y') + d_T(y', y)$ . Now  $d_T(\delta^g, y') \geq d_T(\delta, y) + d_T(y, y')$ , so  $d_T(\delta^g, y) \geq d_T(\delta, y) + 1 > d_T(\delta, y)$ .

Interchanging  $\alpha$  and  $\beta$  in the above argument completes the proof of this lemma.  $\square$

Henceforth, if  $H$  is a subgroup of  $G$ , then we will write  $H \leq G$ ; if we wish to exclude the possibility of  $H = G$  we will instead write  $H < G$ .

**Lemma 2.4.** *Let  $g_1, \dots, g_n \in G_\alpha$  and  $h_1, \dots, h_n \in G_\beta$ , and suppose  $G_{\alpha,y} = G_{\beta,y}$ . If there exists  $\gamma \in VT$  such that  $G_{\alpha,y} \leq G_\gamma$  then, for some  $m \leq n$ , there exist  $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$  and  $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$  together with  $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$  and  $h'_m \in G_\beta \setminus G_y \cup \{1\}$  such that*

$$\gamma^{g'_1 h'_1 \dots g'_m h'_m} = \gamma^{g_1 h_1 \dots g_n h_n}.$$

*Proof.* The proof of this lemma will be an inductive argument. Suppose there exists  $\gamma \in VT$  such that  $G_{\alpha,y} \leq G_\gamma$ .

Let  $n = 1$ . When considering  $h_1 \in G_\beta$  we have two cases: either  $h_1 \in G_y$  or  $h_1 \in G_\beta \setminus G_y$ . If  $h_1 \in G_y$  then  $h_1 \in G_{\beta,y} = G_{\alpha,y}$ , so  $g_1 h_1 \in G_\alpha$ . In this case, redefine  $g_1 := g_1 h_1$  and set  $h'_1 := 1$ . Alternatively, if  $h_1 \in G_\beta \setminus G_y$  then set  $h'_1 := h_1$ . Having found a suitable  $h'_1$ , we will now construct  $g'_1$  from the (possibly redefined) element  $g_1 \in G_\alpha$ . We again have two cases: either  $g_1 \in G_y$  or  $g_1 \in G_\alpha \setminus G_y$ . If  $g_1 \in G_y$  then  $g_1 \in G_{\alpha,y}$  and so  $g_1 \in G_\gamma$ . In this case, we can choose  $g'_1 := 1$ . Otherwise, if  $g_1 \in G_\alpha \setminus G_y$ , then choose  $g'_1 := g_1$ . In choosing  $g'_1$  and  $h'_1$  in this way we ensure

$$\gamma^{g_1 h_1} = \gamma^{g'_1 h'_1},$$

so the hypothesis holds when  $n = 1$ .

Let  $k$  be a positive integer, and suppose the hypothesis is true for all integers  $n \leq k$ . Fix  $g_1, \dots, g_{k+1} \in G_\alpha$  and  $h_1, \dots, h_{k+1} \in G_\beta$ , and set  $\gamma' := \gamma^{g_1 h_1 \dots g_{k+1} h_{k+1}}$ . We will use induction to construct elements  $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$  and  $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$  together with  $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$  and  $h'_m \in G_\beta \setminus G_y \cup \{1\}$  such that

$$\gamma^{g'_1 h'_1 \dots g'_m h'_m} = \gamma',$$

where  $m$  is some integer less than or equal to  $k + 1$ .

We begin by considering  $h_{k+1} \in G_\beta$ . There are two cases: either  $h_{k+1} \in G_y$  or  $h_{k+1} \in G_\beta \setminus G_y$ . If  $h_{k+1} \in G_y$  then  $h_{k+1} \in G_{\beta,y} = G_{\alpha,y}$ , so

$g_{k+1}h_{k+1} \in G_\alpha$ . In this case, redefine  $g_{k+1} := g_{k+1}h_{k+1}$  and set  $h' := 1$ . If, on the other hand,  $h_{k+1} \in G_\beta \setminus G_y$ , then set  $h' := h_{k+1}$ .

If we now consider the (possibly redefined) element  $g_{k+1} \in G_\alpha$ , there are again two cases: either  $g_{k+1} \in G_y$ , or  $g_{k+1} \in G_\alpha \setminus G_y$ . If  $g_{k+1} \in G_y$  then  $g_{k+1} \in G_{\alpha,y} = G_{\beta,y}$ , so  $h_k g_{k+1} h' \in G_\beta$ . In this case, let  $h'' := h_k g_{k+1} h'$ ; then

$$\gamma' = \gamma^{g_1 h_1 \dots g_k h''},$$

so we can apply the induction hypothesis to  $\gamma^{g_1 h_1 \dots g_k h''}$  and we are done.

If, on the other hand,  $g_{k+1} \in G_\alpha \setminus G_y$ , then set  $g' := g_{k+1}$ , and observe

$$\gamma' = \gamma^{g_1 h_1 \dots g_k h_k g' h'}.$$

By the induction hypothesis, for some  $m \leq k$  there exist  $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$  and  $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$  together with  $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$  and  $h'_m \in G_\beta \setminus G_y \cup \{1\}$  such that

$$\gamma^{g_1 h_1 \dots g_k h_k} = \gamma^{g'_1 h'_1 \dots g'_m h'_m}.$$

Set  $g'_{m+1} := g' \in G_\alpha \setminus G_y$  and  $h'_{m+1} := h' \in G_\beta \setminus G_y \cup \{1\}$ . Then

$$\gamma^{g'_1 h'_1 \dots g'_{m+1} h'_{m+1}} = \gamma',$$

so the hypothesis holds for  $n = k + 1$ .  $\square$

We are now in a position to present the main result of this section.

**Theorem 2.5.** *Let  $G$  be a vertex-transitive group of automorphisms of a connectivity-one graph  $\Gamma$  whose blocks have at least three vertices, and let  $T$  be the block-cut-vertex tree of  $\Gamma$ . If there exist distinct vertices  $\alpha, \beta \in V\Gamma$  such that, for some vertices  $\alpha', \beta' \in (\alpha, \beta)_T$ ,*

- (i)  $[\alpha, \alpha']_T \cap (\beta', \beta]_T = \emptyset$ ; and
- (ii)  $G_{\alpha, \alpha'} = G_{\beta, \beta'}$ ;

*then  $G$  does not act primitively on  $V\Gamma$ .*

*Proof.* Suppose  $G$  acts primitively on  $V\Gamma$  and there exist distinct vertices  $\alpha, \beta \in V\Gamma$  and  $\alpha', \beta'$  in the  $T$ -geodesic  $(\alpha, \beta)_T$ , such that (i) and (ii) hold. We will show the group  $\langle G_\alpha, G_\beta \rangle$  generated by  $G_\alpha$  and  $G_\beta$  is not transitive on  $V\Gamma$ . Then  $G_\alpha < \langle G_\alpha, G_\beta \rangle < G$ , which, by applying Theorem 1.1, will contradict the assumption that  $G$  is primitive.

Choose  $y \in [\alpha', \beta']_T$ , and observe that by (ii) we have  $G_{\alpha, y} = G_{\beta, y}$ . Without loss of generality, suppose  $d_T(y, \alpha) \leq d_T(y, \beta)$ . As  $G$  acts primitively on  $V\Gamma$ , the generated group  $\langle G_\alpha, G_\beta \rangle$  is not equal to  $G_\alpha$ , so we must have  $\langle G_\alpha, G_\beta \rangle = G$ . Therefore the orbit  $\beta^{\langle G_\alpha, G_\beta \rangle}$  contains  $\alpha$  and there exist elements  $g_1, \dots, g_n \in G_\alpha$  and  $h_1, \dots, h_n \in G_\beta$  such that  $\alpha = \beta^{g_1 h_1 \dots g_n h_n}$ . By Lemma 2.4, we can find  $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$  and  $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$  together with  $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$  and  $h'_m \in G_\beta \setminus G_y \cup \{1\}$  such that

$$\alpha = \beta^{g'_1 h'_1 \dots g'_m h'_m}.$$

Suppose these automorphisms are chosen so that  $m$  is minimal.

Now either  $g'_1 \in G_\alpha \setminus G_y$  or  $g'_1 = 1$ . If  $g'_1 = 1$  then  $\beta^{g'_1} = \beta$  and therefore  $\beta^{g'_1 h'_1} = \beta$ . Thus  $\beta^{g'_2 h'_2 \dots g'_m h'_m} = \alpha$ , contradicting the minimality of  $m$ . So we must have  $g'_1 \in G_\alpha \setminus G_y$ . Since  $\beta \notin C(T \setminus \{y\}, \alpha)$ , we may apply Lemma 2.2 and Lemma 2.3 to obtain  $d_T(y, \beta^{g'_1}) > d_T(y, \beta)$  and  $\beta^{g'_1} \notin C(T \setminus \{y\}, \beta)$ .

We now observe  $h'_1 \neq 1$ . Indeed, if  $h'_1 = 1$  then  $m = 1$  and  $\alpha = \beta^{g'_1}$ ; since  $g'_1 \in G_\alpha$  this is clearly not possible.

Thus,  $h'_1 \in G_\beta \setminus G_y$  and  $\beta^{g'_1} \notin C(T \setminus \{y\}, \beta)$ , and we can again deduce from Lemma 2.2 and Lemma 2.3 that  $d_T(y, \beta^{g'_1 h'_1}) > d_T(y, \beta^{g'_1}) > d_T(y, \beta)$ , and  $\beta^{g'_1 h'_1} \notin C(T \setminus \{y\}, \alpha)$ .

We may continue to apply Lemmas 2.2 and 2.3 to obtain  $\beta^{g'_1 h'_1 \dots g'_m} \notin C(T \setminus \{y\}, \beta)$  and  $d_T(y, \beta^{g'_1 h'_1 \dots g'_m}) > d_T(y, \beta)$ . Now either  $h'_m \in G_\beta \setminus G_y$  or  $h'_m = 1$ . If  $h'_m = 1$ , then  $\alpha = \beta^{g'_1 h'_1 \dots g'_m}$ , so  $d_T(y, \alpha) = d_T(y, \beta^{g'_1 h'_1 \dots g'_m})$  which is strictly greater than  $d_T(y, \beta)$ . If  $h'_m \in G_\beta \setminus G_y$  then, by Lemma 2.3,  $d_T(y, \beta^{g'_1 h'_1 \dots g'_m h'_m}) > d_T(y, \beta)$ ; that is,  $d_T(y, \alpha) > d_T(y, \beta)$ . Thus, in both cases  $d_T(y, \alpha) > d_T(y, \beta)$ . This contradicts our assumption that  $d_T(y, \alpha) \leq d_T(y, \beta)$ . Hence  $\alpha \notin \beta^{\langle G_\alpha, G_\beta \rangle}$ , and so  $\langle G_\alpha, G_\beta \rangle$  cannot act transitively on the set  $V\Gamma$ .  $\square$

**Theorem 2.6.** *Let  $G$  be a vertex-transitive group of automorphisms of a connectivity-one graph  $\Gamma$  whose blocks have at least three vertices. If  $G$  acts primitively on  $V\Gamma$  and  $\Lambda$  is some block of  $\Gamma$  then  $G_{\{\Lambda\}}$  is primitive and not regular on  $V\Lambda$ .*



*Proof.* Suppose  $\Lambda$  is a block of  $\Gamma$  and  $G_{\{\Lambda\}}$  acts primitively and regularly on  $V\Lambda$ . If  $T$  is the block-cut-vertex tree of  $\Gamma$  then there exists a vertex  $x \in VT$  corresponding to the block  $\Lambda$ . Choose distinct vertices  $\alpha$  and  $\beta$  in  $V\Lambda$ , and observe  $G_{\alpha,x} = G_{\alpha,\{\Lambda\}} \leq G_\beta$  and  $G_{\beta,x} = G_{\beta,\{\Lambda\}} \leq G_\alpha$ . Furthermore,  $x \in (\alpha, \beta)_T$ ; hence  $G$  cannot act primitively on  $VT$  by Theorem 2.5.  $\square$

### 3. GLOBAL STRUCTURE

In this section we will employ Theorem 2.6 to give a complete characterisation of the primitive connectivity-one directed graphs.

**Lemma 3.1.** *Suppose  $\Gamma$  is a vertex-transitive graph with connectivity one, whose blocks are vertex-transitive, have at least three vertices and are pairwise isomorphic. If  $\Lambda$  is a block of  $\Gamma$  and  $H$  is the subgroup of  $\text{Aut } \Lambda$  induced by the action of  $(\text{Aut } \Gamma)_{\{\Lambda\}}$  on  $\Lambda$ , then  $H = \text{Aut } \Lambda$ .*

*Proof.* Let  $T$  denote the block-cut-vertex tree of  $\Gamma$ , and let  $\Lambda$  be a block of  $\Gamma$ . We will show any automorphism of the directed graph  $\Lambda$  may be extended to an automorphism of  $\Gamma$ .

We begin by asserting that if  $\Lambda_1$  and  $\Lambda_2$  are blocks of  $\Gamma$ , and  $\alpha_1$  and  $\alpha_2$  are vertices in  $\Lambda_1$  and  $\Lambda_2$  respectively, then there exists an isomorphism  $\rho : \Lambda_1 \rightarrow \Lambda_2$  such that  $\alpha_1^\rho = \alpha_2$ . Indeed, by assumption, there exists an isomorphism  $\rho' : \Lambda_1 \rightarrow \Lambda_2$ . Define  $\alpha'_1 := \alpha_1^{\rho'}$ . Since the block  $\Lambda_2$  is vertex-transitive, there exists an automorphism  $\tau$  of  $\Lambda_2$  such that  $\alpha'_1{}^\tau = \alpha_2$ . Let  $\rho := \rho'\tau$ . Then  $\rho : \Lambda_1 \rightarrow \Lambda_2$  is an isomorphism, with  $\alpha_1^\rho = \alpha_1^{\rho'\tau} = \alpha'_1{}^\tau = \alpha_2$ .

Let  $x$  be the vertex of  $T$  that corresponds to  $\Lambda$ . For  $k \geq 0$ , define  $\Gamma_k$  to be the subgraph of  $\Gamma$  induced by the set  $\{\alpha \in VT \mid d_T(x, \alpha) \leq 2k + 1\}$ . We will show any automorphism  $\sigma_k : \Gamma_k \rightarrow \Gamma_k$  admits an extension  $\sigma_{k+1} : \Gamma_{k+1} \rightarrow \Gamma_{k+1}$ . Whence, by induction, the lemma will follow.

Fix  $k \geq 0$  and let  $\sigma_k : \Gamma_k \rightarrow \Gamma_k$  be an automorphism. Let  $\{\alpha_i\}_{i \in I}$  be the set of vertices in  $V\Gamma_k \setminus V\Gamma_{k-1}$  (where  $V\Gamma_{-1} := \emptyset$ ). Each vertex  $\alpha_i$  belongs to a unique block  $\Lambda_i$  of  $\Gamma_k$ , and, if  $k \geq 1$ , the block  $\Lambda_i$  possesses exactly one vertex in  $\Gamma_{k-1}$ . Since  $\Gamma$  is vertex transitive, any two vertices lie in the same number of blocks of  $\Gamma$ , so let  $\{\Lambda_{i,j}\}_{j \in J}$  be the set of

blocks of  $\Gamma$  that contain  $\alpha_i$  and are distinct from  $\Lambda_i$ . Each block  $\Lambda_{i,j}$  is wholly contained in  $\Gamma_{k+1}$  and has exactly one vertex in  $\Gamma_k$ , namely  $\alpha_i$ . If  $i \in I$ , set  $\alpha'_i := \alpha_i^{\sigma_k}$  and  $\Lambda'_i := \Lambda_i^{\sigma_k}$ . Then  $\Lambda'_i = \Lambda_{i'}$  for some  $i' \in I$ . For all  $j \in J$  there exists an isomorphism  $\rho_{i,j} : \Lambda_{i,j} \rightarrow \Lambda_{i',j}$  such that  $\alpha_i^{\rho_{i,j}} = \alpha'_{i'}$ . Thus, we may define a mapping  $\sigma_{k+1} : \Gamma_{k+1} \rightarrow \Gamma_{k+1}$  with

$$\beta^{\sigma_{k+1}} := \begin{cases} \beta^{\sigma_k} & \text{if } \beta \in V\Gamma_k; \\ \beta^{\rho_{i,j}} & \text{if } \beta \in V\Lambda_{i,j}. \end{cases}$$

This is clearly a well-defined automorphism of  $\Gamma_{k+1}$ .  $\square$

The primitive undirected graphs with connectivity one have the following complete characterisation.

**Theorem 3.2.** ([3, Theorem 4.2]) *If  $\Gamma$  is a vertex-transitive undirected graph with connectivity one, then it is primitive if and only if the blocks of  $\Gamma$  are primitive, pairwise isomorphic and each has at least three vertices.*  $\square$

This useful result seems to suggest a similar characterisation may be possible for directed primitive graphs with connectivity one. This is indeed the case.

**Theorem 3.3.** *If  $\Gamma$  is a vertex-transitive directed graph with connectivity one, then it is primitive if and only if the blocks of  $\Gamma$  are primitive but not automorphism-regular, pairwise isomorphic and each has at least three vertices.*

*Proof.* Let  $\Gamma$  be a directed vertex-transitive graph with connectivity one. Suppose the blocks of  $\Gamma$  are primitive but not automorphism-regular, pairwise isomorphic and each has at least three vertices. Let  $\approx$  be a non-trivial  $\text{Aut } \Gamma$ -congruence on  $V\Gamma$ . We will show this relation must be universal, and thus that  $\Gamma$  is a primitive graph. Since the relation is non-trivial, there exist distinct vertices  $\alpha, \beta \in V\Gamma$  such that  $\alpha \approx \beta$ . Let  $T$  be the block-cut-vertex tree of  $\Gamma$ , let  $\gamma \in V\Gamma$  be the vertex in the geodesic  $[\alpha, \beta]_T$  such that  $d_T(\beta, \gamma) = 2$ , and let  $\Lambda$  be the block of  $\Gamma$  containing  $\beta$  and  $\gamma$ . By Lemma 3.1, the group  $(\text{Aut } \Gamma)_{\{\Lambda\}}$  acts primitively but not regularly on  $V\Lambda$ . Thus, there exists an automorphism  $g \in (\text{Aut } \Gamma)_{\gamma, \{\Lambda\}}$  that does not fix  $\beta$ . We are considering

the full automorphism group of the connectivity-one graph  $\Gamma$ , so there must therefore exist an element  $g' \in (\text{Aut } \Gamma)_{\alpha, \gamma, \{\Lambda\}}$  that does not fix  $\beta$ . Thus,  $\beta$  and  $\beta^{g'}$  are distinct vertices in  $\Lambda$ . Now  $\alpha \approx \beta$ , so  $\alpha \approx \beta^{g'}$ , and therefore  $\beta \approx \beta^{g'}$ . Since  $(\text{Aut } \Gamma)_{\{\Lambda\}}$  is primitive on  $V\Lambda$  and  $\approx$  induces a non-trivial  $(\text{Aut } \Gamma)_{\{\Lambda\}}$ -congruence on  $V\Lambda$ , this relation must be universal in  $\Lambda$ . By assumption,  $\text{Aut } \Gamma$  acts transitively on the blocks of  $\Gamma$ , so if two vertices lie in the same block then they must lie in the same congruence class. Thus, if  $\gamma$  is any vertex of  $\Gamma$ , and  $\alpha x_1 \alpha_1 x_2 \dots x_n \gamma$  is the geodesic in  $T$  between  $\alpha$  and  $\gamma$ , then  $\alpha$  and  $\alpha_1$  lie in a common block, so  $\alpha \approx \alpha_1$ . Similarly,  $\alpha_1 \approx \alpha_2$  and  $\alpha_2 \approx \alpha_3$ , so  $\alpha \approx \alpha_2$  and  $\alpha \approx \alpha_3$ . Continuing in this way we eventually obtain  $\alpha \approx \gamma$ . Hence, this congruence relation is universal on  $V\Gamma$ .

Conversely, suppose the group  $\text{Aut } \Gamma$  acts primitively on  $V\Gamma$ . Since  $\Gamma$  is a directed primitive graph with connectivity one, we can obtain an undirected graph  $\Gamma'$  with vertex set  $V\Gamma$  and edge set  $\{\{\alpha, \beta\} \mid (\alpha, \beta) \in E\Gamma\}$ . Two vertices are adjacent in  $\Gamma$  if and only if they are adjacent in  $\Gamma'$ . As  $\text{Aut } \Gamma$  is primitive on  $V\Gamma$  and  $\text{Aut } \Gamma \leq \text{Aut } \Gamma'$ , it follows that  $\text{Aut } \Gamma'$  must be primitive on  $V\Gamma$ , and hence  $\Gamma'$  is a primitive undirected graph. Since  $\Gamma$  has connectivity one, the same is true of  $\Gamma'$ , so we may apply Theorem 3.2 to deduce the blocks of  $\Gamma'$  are primitive, pairwise isomorphic and each has at least three vertices. Now, given a block  $\Lambda$  of  $\Gamma$ , there is a block  $\Lambda'$  of  $\Gamma'$  such that  $V\Lambda = V\Lambda'$ . Therefore, the blocks of  $\Gamma$  have at least three vertices, and are primitive but not automorphism-regular by Theorem 2.6.

It remains to show they are pairwise isomorphic. Fix some block  $\Lambda$  of  $\Gamma$  and an edge  $(\alpha, \beta) \in E\Lambda$ . Let  $\Gamma_1$  be the graph  $(V\Gamma, (\alpha, \beta)^{\text{Aut } \Gamma})$ . As  $\text{Aut } \Gamma$  is primitive, this graph is a connected subgraph of  $\Gamma$ . Thus, every block of  $\Gamma$  must contain an edge in  $E\Gamma_1$ . Furthermore, if  $\Lambda'$  is a block of  $\Gamma$ , then any automorphism of  $\Gamma$  mapping the edge  $(\alpha, \beta)$  to an edge in  $\Lambda'$  must map  $\Lambda$  to  $\Lambda'$ . Since  $\Gamma_1$  is edge-transitive, the blocks of  $\Gamma$  must be pairwise isomorphic.  $\square$

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